A METHOD OF TAKING ACCOUNT OF *A PRIORI* INFORMATION IN SOLVING INCORRECT INVERSE PROBLEMS

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An approach to taking into account *a priori* information about the smoothness of the function being restored is elucidated, as are also values of the function and its derivatives at a number of points of the domain of definition.

The confidence in and accuracy of the solution of incorrectly posed problems can depend to a significant extent on how completely available information about the quantities desired is taken into account. This information is separated into qualitative and quantitative. In the former case the presence of information about the smoothness of the functions being reproduced is understood, and certain quantitative characteristics of these functions to the latter. Such information can be given by starting from the physical singularities of the processes under investigation and the features of conducting the experiment as well as from conditions of uniqueness in solving the problem.

The iteration form of regularization [7-14] provides sufficiently broad possibilities for taking account of the qualitative and quantitative *a priori* information about the solution of the incorrect problem. One such approach, according to which the direction of descent in the construction of the iteration sequence corresponding to the gradient method is selected in the initial space of solutions $U = L_2$ (integrable square functions) so as not to deduce approximations from the class W_2^k (functions having k generalized derivatives), is considered in [7, 15, 16]. Another approach proposed in [17], when the iteration sequence is obtained directly in the $U = W_2^k$, is developed in the present paper.

1. TAKING ACCOUNT OF QUALITATIVE INFORMATION ABOUT THE SMOOTHNESS OF THE DESIRED SOLUTION

Let us consider an operator equation of the first kind

$$Au = f, \tag{1}$$

where $A: W_2^k[a, b] \to L_2[a, b]$; $u = u(x)$, $x \in R^1$, is the desired function, and $f = f(x)$ is given.

We will consider the solution of the problem (1) to exist and be unique, but correct solvability of this equation is spoiled, the inverse operator A^{-1} is not continuous.

Let the right side of (1) be given with the error $f_\delta = f + \tilde{f}$, $\|\tilde{f}\|_F \leq \delta$.

We construct an algorithm of the solution of (1) on the basis of a certain approximation process

$$u^{j+1} = F_A(u^j, \delta), \quad j = 0, 1, \ldots, j^*,$$

in which the number j of the iteration is considered as the regularization parameter. In particular, such an iteration can correspond to steepest descent and conjugate gradient methods. It is established in [12, 13] for the linear case that these methods generate a family of regularizing parameters with parameter j. If the iteration process is set up according to the residual criterion $(j^*: \|Au^{j^*} - f_\delta\|_F \simeq \delta)$, then the methods mentioned are regularizing algorithms (more accurate formulations are presented in the above-mentioned papers).

The applicability of such an approach to the solution of a number of incorrect problems in a nonlinear formulation was shown by the method of numerical modelling [7, 9, 18-21]. Consequently, in the general case we examine (1) with the nonlinear operator A which we will consider Frechet differentiable.

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We later examine the algorithm to solve the problem (1) by the appropriate method of steepest descent (we easily realize the transition to the conjugate gradient method). In this case we have the iteration sequence

$$u^{j+1} = u^j - \beta_j J'^{j}_{W_2^k}, \quad j = 0,\ 1,\ \ldots,\ j^*; \tag{2}$$

$$J'^{j}_{W_2^k} = 2\,(A'_{u^j})^*\,(Au^j - f_\delta),$$

where $J'_{W_2^k}$ is the gradient in u for the functional-residual $J(u) = \|Au - f_\delta\|^2_{L_2}$ in the space W_2^k, A_u' is the Frechet derivative of the operator A, and β_j is the step in the descent to the j-th iteration $(\beta_j : \min\limits_{\beta} J(u^j - \beta J'^{j}_{W_2^k}))$.

The initial approximation u^0 in (2) must be selected from the class of functions of appropriate smoothness $u^0(x) \in W_2^p[a,\ b]$, $p \geqslant k$, in particular, we can set $u^0(x) = 0$.

When using (2) and selecting the number of the last iteration j^* by means of the residual criterion, we have $u^{j^*} \xrightarrow[W_2^k]{} \bar{u}$ as $\delta \to 0$, where $\bar{u} \in W_2^k$ is the exact solution of the problem (1).

Therefore, for the practical application of this algorithm, a method must be found to determine the gradient of the functional J(u) in the space W_2^k. We shall consider that there is an algorithm to find the gradient in the space $L_2[a,\ b]$ at our disposal, which will be denoted by J'_{L_2}. In particular, such algorithms can be constructed by using an adjoint boundary value problem [7, 17, 18-23] when solving a broad circle of inverse problems and optimal control problems for systems with lumped and distributed parameters.

It is shown in [17] that the following boundary value problem

$$\sum_{n=0}^{k} (-1)^n \frac{d^n}{dx^n}\left(r_n \frac{d^n J'_{W_2^k}}{dx^n}\right) = J'_{L_2}(x), \quad x \in (a,\ b); \tag{3}$$

$$\sum_{i=n}^{k} (-1)^{i+1} \frac{d^{i-n}}{dx^{i-n}}\left(r_n \frac{d^i J'_{W_2^k}}{dx^i}\right)\Bigg|_{x=l} = 0, \quad n = \overline{1,\ k};\ l = a,\ b. \tag{4}$$

must be solved to determine the gradient $J'_{W_2^k}$ by means of a given gradient J'_{L_2}. Here $r_n = r_n(x)$ are given nonnegative continuous functions that play the part of weights, where r_0, $r_k > 0$. The influence of each of the derivatives on the desired function u(x) at different points of the segment $[a,\ b]$ is taken into account with their aid. It is ordinarily assumed that r_n are numerical factors, for instance $r_1 = r_2 = \ldots = r_k = 1$.

In the case $u \in W_2^1$, which turns out to be perfectly suitable for many practical applications, the boundary value problem (3), (4) takes on its simplest form

$$r_0 J'_{W_2^1} - \frac{d}{dx}\left(r_1 \frac{dJ'_{W_2^1}}{dx}\right) = J'_{L_2},\ x \in (a,\ b); \tag{5}$$

$$\frac{dJ'_{W_2^1}}{dx}\Bigg|_{x=a} = \frac{dJ'_{W_2^1}}{dx}\Bigg|_{x=b} = 0. \tag{6}$$

Assuming that r_0 and r_1 are numbers, the solution of the problem (5), (6) can be obtained in terms of the Green's function

$$J'_{W_2^1}(x) = B_1 \exp[\rho x] + B_2 \exp[-\rho x] - \frac{1}{\rho r_1}\int\limits_a^x J'_{L_2}(\xi)\,\mathrm{sh}\,[\rho(x-\xi)]\,d\xi,\ x \in [a,\ b], \tag{7}$$

where

$$\rho = \sqrt{\frac{r_0}{r_1}};\ B_1 = B_2 \exp[-2\rho a];$$

$$B_2 = \frac{\int\limits_a^b J'_{L_2}(\xi)\,\mathrm{ch}\,[\rho(b-\xi)]\,d\xi}{\rho r_1(\exp[\rho b - 2\rho a] - \exp[\rho b - 2\rho b])}.$$

Let us note that it is expedient to take account of the possibility of analytic integration of the function sinh z in evaluating the integral in (7). In particular, a sufficiently effective procedure for computing $\int\limits_a^x \ldots d\xi$ is obtained when using the simplest step approximation of the function

$$\int\limits_a^x J'_{L_2}(\xi)\,\mathrm{sh}\,[\rho(x-\xi)]\,d\xi \simeq \sum_{i=1}^{n} J'_i \int\limits_{x_{i-1}}^{x_i} \mathrm{sh}\,[\rho(x_n-\xi)]\,d\xi = -\rho \sum_{i=1}^{n} J'_i(\mathrm{ch}\,[\rho(x_n-x_i)] - \mathrm{ch}\,[\rho(x_n-x_{i-1})]),$$

where

$$J'_i = J'_{L_2}\left(x_i - \frac{x_i - x_{i-1}}{2}\right).$$

The integral in the expression for the constant B_2 can be calculated by an analogous method.

2. TAKING ACCOUNT OF QUANTITATIVE INFORMATION ABOUT THE SOLUTION

We first examine the situation when values of the function and (or) its derivatives are known at the boundary points of the segment $[a, b]$. These data can be taken into account sufficiently simply by the selection of the initial approximation to the solution and by giving appropriate boundary conditions for (3). For instance, let the values of the derivatives $u'(a) = \alpha_1$ and $u'(b) = \beta_1$ be known. We take this information into account in selecting the initial approximation, namely, we require satisfaction of the equalities $\frac{du^0(a)}{dx} = \alpha_1$, $\frac{du^0(b)}{dx} = \beta_1$. Now, if the solution (7) of the problem (5), (6) is used, then the conditions mentioned will be satisfied exactly by virtue of the equalities (6). When the values of the functions $u(a) = \alpha_0$, $u(b) = \beta_0$ are known, then can be taken into account when using the space W_2^1 by replacing the boundary conditions (6) by others: $J'_{W_2^1}(a) = J'_{W_2^1}(b) = 0$. The initial approximation is given here by conserving the equalities $u^0(a) = \alpha_0$, $u^0(b) = \beta_0$. Taking simultaneous account of the values of the functions and the first derivative at the edges of the segment is possible in solving the problem (1) in the space W_2^2. In this case it is necessary to give $\frac{d^n J'_{W_2^2}(l)}{dx^n} = 0,\ l = a, b,\ n = 1, 2,$ in place of the conditions (4) and to select $u^0(x)$ in an appropriate manner. Within the framework of the conditions (4), taking account of the values of the second derivative on the segment boundaries is possible for $k = 2$. Other cases giving the *a priori* information about the desired solution at the points a and b can also be considered analogously.

The method being considered for the construction of a smooth solution permits taking account also of certain *a priori* information about the function and its derivatives at a number of points of the segment $[a, b]$. We turn to a clarification of these questions. Functions of the class $W_2^k[a, b]$ can be represented in the following integral form in terms of the generalized derivative $u^{(k)}(x)$ [16]:

$$u(x)=\sum_{n=0}^{k-1}C_nP_n(x)+\int_{x_1}^{x}d\xi_1\int_{x_2}^{\xi_1}d\xi_2\ldots\int_{x_k}^{\xi_{k-1}}u^{(k)}(\xi_k)\,d\xi_k, \tag{8}$$

where $C_n = u^{(n)}(x_{n+1})$, $n=\overline{0, k-1}$ are values of the derivatives of the function u(x) to order $k-1$ at certain fixed points $x_{n+1}\in[a, b]$; $P_n(x)$ is a polynomial of n-th degree.

The identity (8) is obtained as follows. If the function $u(x)\in L_2[a, b]$ has a generalized k-th derivative $u^{(k)}(x)\ L_2[a, b]$ then it is continuously differentiable $k-1$ times in the segment [a, b] and the derivative $u^{(k-1)}(x)$ is absolutely continuous in $[a, b]$. In this case, the relationship

$$u^{(n-1)}(\xi_{n-1})=\int_{x_n}^{\xi_{n-1}}u^{(n)}(\xi_n)\,d\xi_n+u^{(n-1)}(x_n),\ \ n=\overline{1,k},$$

holds for all derivatives to order $k-1$.

Expressing u(x) in terms of $u'(\xi_1)$, then $u'(\xi_1)$ in terms of $u''(\xi_2)$, etc., we arrive at the identity (8). The form of the polynomials $P_n(x)$ is obtained easily for each specific problem.

Let us use the notation $y(x)=\int_{x_1}^{x}d\xi_1\int_{x_2}^{\xi_1}d\xi_2\ldots\int_{x_k}^{\xi_{k-1}}u^{(k)}(\xi_k)\,d\xi_k$ and let us substitute u(x) in the form (8) into the iteration sequence (2). We consequently have

$$y^{j+1}(x)+\sum_{n=0}^{k-1}C_n^{j+1}P_n(x)=y^j(x)+\sum_{n=0}^{k-1}C_n^jP_n(x)-\beta_jJ'_{W_2^k}. \tag{9}$$

Furthermore, we assume that values of the function and its derivatives are known at the points $\{x_n\}$, i.e., the numbers C_n, $n=\overline{0, k-1}$ are given. In this case (9) takes the form

$$y^{j+1}(x)=y^j(x)-\beta_jJ'^{j}_{W_2^k},$$

where the initial approximation $y^0(x)$ should correspond to the conditions $u^{0(n)}(x_{n+1})=C_n$, $n=\overline{0, k-1}$, in particular, it can be assumed that $u^0(x)=\sum_{n=0}^{k-1}C_nP_n(x)$, then $y^0(x)=0$.

By this method k conditions in the form of an equality for the function $u(x_1)$ itself and its derivatives $u^{(n)}(x_{n+1})$, $n=\overline{0, k-1}$ can be satisfied exactly, each of these quantities is satisfied at one of the points of the segment $[a, b]$ (these points can also certainly coincide). Hence, it becomes clear how to select the values of x_n, $n=\overline{0, k-1}$.

The method described permits taking account of one value of the function and by means of one value of each derivative at certain points of the segment $[a, b]$, including at its boundary. If a large number of values of both the function and its derivatives should be satisfied, we can then proceed as follows. The interval $[a, b]$ is partitioned into subdomains whose boundaries agree with the points x_n where conditions are given and s + 1 boundary value problems of the form (3)-(4) are solved in appropriate domains ($[a, x_1]$; $[x_n, x_{n+1}]$, $n=\overline{1, s-1}$; $[x_s, b]$). Let us note that the gradients J'_{L_2} are determined by individual sections of the segment $[a, b]$. It is natural that the initial approximation $u^0(x)$, $x\in[a, b]$ should be selected in agreement with the given quantities $u^{(i)}(x_n)$.

3. PARAMETRIZED MODE OF THE SOLUTION

In solving a number of inverse problems and optimal control problems, it turns out to be convenient to represent the desired function in the following approximate form:

$$u(x)\simeq\tilde{u}(x)=\sum_{\eta=1}^{M}a_\eta\varphi_\eta(x),\quad x\in[a, b], \tag{10}$$

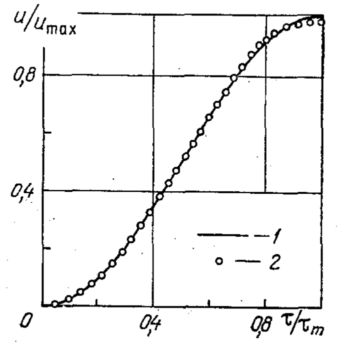

Fig. 1. Results of recovery of the heat flux density as a function of the time 1) desired solution; 2) compared data.

where $\{\varphi_\eta(x)\}_1^M$ is a given system of basis functions, and $\mathbf{a}=\{a_\eta\}_1^M$ is the numerical vector of the coefficients to be determined.

Cubic B-splines are often taken as $\varphi_\eta(x)$. The corresponding functions (10) form a subspace in the space $W_2^k[a, b]$, $k\leqslant 3$.

To determine $\mathbf{a} \in R^M$ we use an iteration formula of gradient type with a halt in the residual

$$\mathbf{a}^{j+1}=\mathbf{a}^j-\beta_j\mathbf{p}^j,\quad j=0,\ 1,\ \ldots,\ j^*,$$
$$j^*: J(\mathbf{a}^{j^*})=\|A\tilde{u}(x)-f_\delta\|_{L_2}^2\simeq\delta^2. \tag{11}$$

Let us pose the problem: Find the descent direction $\mathbf{p}^j=\{p_\eta^j\}_{\eta=1}^{\eta=M}$ in each iteration in such a manner as to assure convergence of the approximation $\tilde{u}^j(x)$ in the norm of the space W_2^k. For simplicity, we limit ourselves, as before, to the method of steepest descent in this analysis.

Let the desired function $\tilde{u}(x) \in W_2^k[a, b]$ receive a small increment $\theta(x) \in W_2^k[a, b]$. Then the linear part of the appropriate increment in the functional J can be represented in the form of a scalar product in the space $W_2^k[a, b]$:

$$\Delta J=(\theta,\ J'_{W_2^k})_{W_2^k}\equiv\sum_{n=0}^{k}\int_a^b r_n\frac{d^n J'_{W_2^k}}{dx^n}\frac{d^n\theta}{dx^n}dx. \tag{12}$$

Since $\tilde{u}(x)$ has the form (10), we obtain for the increment $\theta(x)$

$$\theta(x)=\sum_{\eta=1}^{M}\Delta a_\eta\varphi_\eta(x). \tag{13}$$

Taking into account that the vector p in the interation sequence (11) corresponds to the method of steepest descent, we write an analogous representation for the gradient of the functional in the space W_2^k

$$J'_{W_2^k}=\sum_{\eta=1}^{M}p_\eta\varphi_\eta(x). \tag{14}$$

After substituting (13) and (14) into (12) and some manipulations, we arrive at the following expression

$$\Delta J=(\Delta\mathbf{a},\ \mathbf{v})_{R^M}, \tag{15}$$

where

$$\Delta \mathbf{a} = \{\Delta a_\eta\}_1^M;\ \mathbf{v} = \{v_\eta\}_1^M,\ v_\eta = \sum_{i=1}^{M} p_i (\varphi_\eta,\ \varphi_i)_{W_2^k};\ (\ ,\ )_{R^M}$$

and $(,)_{R^M}$ is the scalar product in the space R^M.

It follows from (15) that the vector **v** is a gradient of the functional J in the space R^M, from which we obtain a system of linear algebraic equations to calculate the components p_η, if the components v_η are known:

$$\sum_{i=1}^{M} p_i b_{\eta i} = v_\eta,\ \eta = \overline{1,\ M}, \quad (16)$$

where $b_{\eta i} = (\varphi_\eta,\ \varphi_i)_{W_2^k}$.

The matrix of this system is symmetric, and positive-definite, and methods that take account of these features and are well known in linear algebra can be used to solve (16).

The right side of the system (16) is the gradient of a functional in **a**, it can be found in terms of the solution of the adjoint boundary value problem, as is done in particular, in [19, 20].

The elucidated approach to take account of qualitative and quantative information permits the construction of effective computational algorithms and yields good results in solving different practical problems. A graph of the solution of the inverse boundary-value heat-conduction problem with constant coefficients is shown in the figure for an example. That the desired function $u(\tau)$ belongs to the space $W_2^1[0,\ \tau_m]$ was given as information about the smoothness and values of the derivatives $u'(0) = u'(\tau_m) = 0$ were known on the boundaries of a time interval. The quantity $u^0(\tau)$ was taken as initial approximation. The gradient of the functional-residual was computed by means of (7) by using the step approximation $u(\tau)$. The initial data were taken unperturbed. As is seen from the graph, restoration of the curve is close to the exact value.

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WAYS OF ALLOWING FOR *A PRIORI* INFORMATION IN REGULARIZING GRADIENT ALGORITHMS

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Ways of allowing for *a priori* information on an unknown quantity in the solution of boundary-value and coefficient inverse problems of heat conduction by gradient methods are considered.

In the solution of inverse problems of heat conduction (IPHC), like any other ill-posed problem the qualitatively obtained approximations essentially depend on the proper and complete allowance for all the available *a priori* information about the solution being sought [1, 2]. And the widespread case in IPHC is the presence of information about the smoothness of the solution.

Let an IPHC be formulated as an operator equation of the first kind,

$$Au = f, \quad u \in U, \quad f \in F, \tag{1}$$

where we shall take the operator A as Frechet differentiable. The choice of the spaces U and F is dictated by the statement of the problem itself: They must contain sufficiently broad classes of functions, which will include all physically possible solutions u and any initial data f with allowance for the distortions introduced by the measurement systems. Therefore, the space L_2 of functions with an integrable square is taken most often as the spaces U and F. This is a Hilbert space, enabling one to apply gradient methods for the solution of Eq. (1).

For concrete problems, however, there is often additional, qualitative, *a priori* information about the solution being sought, which is usually given in one of two forms:

1) $u \in L[V]$, a transform of a certain continuous linear operator $L: V \to U$;

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